Maximal subgroups of free idempotent generated semigroups

York Semigroup
28th January 2015

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Or.....
Groups and Idempotents

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Semigroups and monoids

Throughout, S is a **semigroup** i.e. a set with an associative binary operation.

If $\exists\, 1 \in S$ with $1a = a = a1$ for all $a \in S$, then S is a **monoid**.

An **idempotent** is an element $e \in S$ such that $e = e^2$.

Let $E = \{e \in S : e \text{ is idempotent}\} = E(S)$.

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Examples of semigroups

- Groups
- Multiplicative semigroups of rings, e.g. $M_n(\mathbb{R})$
- Let X be a set:

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\begin{aligned}
\mathcal{T}_X : &= \{\alpha \mid \alpha : X \to X\} \\
\mathcal{PT}_X : &= \{\alpha \mid \alpha : Y \to Z, Y, Z \subseteq X\} \\
\mathcal{S}_X : &= \{\alpha \mid \alpha : X \to X, \alpha \text{ bijective}\} \\
\mathcal{I}_X : &= \{\alpha \mid \alpha : Y \to Z, Y, Z \subseteq X, \alpha \text{ one-one}\}
\end{aligned}
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  are monoids under $\circ$, the **full transformation monoid**, the **partial transformation monoid**, the **symmetric group**, and the **symmetric inverse monoid** on X, respectively.
- If $X = \underline{n} = \{1, \ldots, n\}$, then we usually write $\mathcal{T}_n$ for $\mathcal{T}_X$, etc.

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Examples of semigroups: free semigroups

- Let X be a set and let

$$X^+ = \{x_1 \ldots x_n : n \in \mathbb{N}\}$$

with

$$(x_1 \ldots x_n)(y_1 \ldots y_m) = x_1 \ldots x_n y_1 \ldots y_m.$$

Then X^+ is the **free semigroup** on X.

Monoids of endomorphisms

Let A be set with structure e.g. A is partially ordered, or A is an **algebra** (in the sense of universal algebra).

Let

$$\operatorname{End} A = \{\alpha \mid \alpha : A \to A \text{ preserves the structure}\}.$$

Then $\operatorname{End} A$ is a monoid, the **endomorphism monoid** of A.

Bands

A **band** is a semigroup S such that $S = E(S)$.

Let I, Λ be non-empty sets, let $T = I \times \Lambda$ and define

$$(i, \lambda)(k, \mu) = (i, \mu).$$

Then T is a band, the **rectangular band** on $I \times \Lambda$.

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Idempotent generated semigroups

For a subset A of a semigroup S we put

$$\langle A \rangle = \{a_1 \dots a_n : n \geq 1,\ a_i \in A\};$$

$\langle A \rangle$ is the subsemigroup **generated by** A.

Clearly X^+ is generated by X.

S is **idempotent generated** if $S = \langle E \rangle$.

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Examples of idempotent generated semigroups

Some familiar semigroups are idempotent generated:

Example 1 For $\alpha \in \mathcal{T}_n$ let **rank** $\alpha = |\operatorname{Im}\alpha|$. Clearly

$$\mathcal{S}_n = \{\alpha \in \mathcal{T}_n : \operatorname{rank}\alpha = n\}.$$

Let

$$S(\mathcal{T}_n) = \{\alpha \in \mathcal{T}_n \mid \operatorname{rank}\alpha < n\}.$$

Then $S(\mathcal{T}_n)$ is idempotent generated (**Howie, 1966**).

$S(\mathcal{T}_n)$ is called the **singular part** of $\mathcal{T}_n$ and often denoted Sing_n.

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Examples of idempotent generated semigroups

Example 2 Let D be a division ring. Then

$$S(M_n(D)) = \{A \in M_n(D) : \operatorname{rank} A < n\}$$

is idempotent generated (**J.A. Erdös, 1967, Dawlings, 1979, Laffey, 1973**).

Example 3 Let A be an independence algebra with rank $A = n$. Then

$$S(\operatorname{End} A) = \{\alpha \in \operatorname{End}(A) : \operatorname{rank} \alpha < n\}$$

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Green's relations: $\mathcal{R}, \mathcal{L}, \mathcal{H}$ and $\mathcal{D}$

Let S be a semigroup. Then

$$\begin{aligned}
a\,\mathcal{R}\,b &\Leftrightarrow aS^1 = bS^1 \Leftrightarrow a = bs,\ b = at,\ s,t \in S^1 \\
a\,\mathcal{L}\,b &\Leftrightarrow S^1a = S^1b \Leftrightarrow a = sb,\ b = ta,\ s,t \in S^1 \\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R} \\
\mathcal{D} &= \mathcal{L} \vee \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.
\end{aligned}$$

A trick to note For $e, f \in E$, if $eS^1 \subseteq fS^1$, then from $e = fu$ we obtain

$$fe = ffu = fu = e.$$

Hence

$$\begin{aligned}
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A curious thought

Let G be a group. How is G related to idempotents?

A group has precisely one idempotent, thus G **is idempotent generated if and only if G is trivial.**

A **subgroup** of S is a subsemigroup of S that is a group under the restricted product.

Fact The maximal subgroups of S are the $\mathcal{H}$-classes

$$H_e, e \in E.$$

Is it possible that G is a subgroup of S such that

$$G \subseteq \langle E \rangle?$$

Yes! Let $|G| = r < \infty$. Pick $\epsilon = \epsilon^2 \in S = S(\mathcal{T}_n)$ with rank $\epsilon = r$; it is known that $H_\epsilon \cong \mathcal{S}_r$. Then

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Free G-acts

Let G be a group and $n \in \mathbb{N}$.

The **free** G**-act** $F_n(G)$ is given by

$$F_n(G) = Gx_1 \cup Gx_2 \cup \ldots \cup Gx_n$$

where

$$g(hx_i) = (gh)x_i \text{ (note we identify } x_i \text{ with } ex_i).$$

If $\alpha \in \operatorname{End} F_n(G)$ then $(gx_i)\alpha = g(x_i\alpha)$ and we can write $\alpha \in \operatorname{End} F_n(G)$ as

$$\alpha = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ g_1 x_{1\overline{\alpha}} & g_2 x_{2\overline{\alpha}} & \cdots & g_n x_{n\overline{\alpha}} \end{pmatrix}.$$

Note that $F_n(G)$ is an **independence algebra**, so

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Free G-acts: rank 1 $\mathcal{D}$-class

For elements $\alpha, \beta \in \operatorname{End} F_n(G)$ or $S = S(\operatorname{End} F_n(G))$ we have

$$\begin{aligned} \alpha \,\mathcal{R}\, \beta &\Leftrightarrow \operatorname{Ker}\alpha = \operatorname{Ker}\beta \\ \alpha \,\mathcal{L}\, \beta &\Leftrightarrow \operatorname{Im}\alpha = \operatorname{Im}\beta \\ \alpha \,\mathcal{D}\, \beta &\Leftrightarrow \operatorname{rank}\alpha = \operatorname{rank}\beta. \end{aligned}$$

Let $D_1 = \{\alpha \in S : \operatorname{rank}\alpha = 1\}$.

Fact D_1 is **completely simple.** That is,

$$D_1 = \bigcup_{(i,\lambda) \in I \times \Lambda} H_{i\lambda}$$

and each $H_{i\lambda}$ is a group; moreover,

$$H_{i\lambda} H_{j\mu} \subseteq H_{i\mu}.$$

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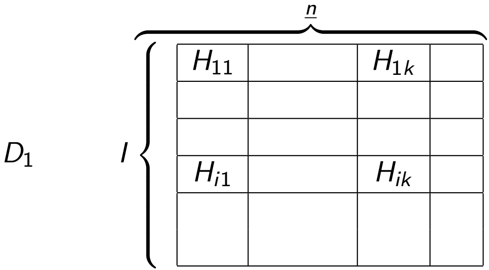
$\underline{n}$ indexes the $\mathcal{L}$-classes L_k

$$\alpha \in L_k \Leftrightarrow \operatorname{Im} \alpha = Gx_k.$$

I indexes $\mathcal{R}$-classes R_i

R_1 corresponds to the kernel $\langle (x_1, x_i) : 1 \leq i \leq n \rangle$

Put $H_{i\lambda} = R_i \cap L_\lambda$.



It is known that $H_{11} \cong H_{i\lambda} \cong G$, for any i, λ, so again,

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D_1 $\quad I$ (rows), $\underline{n}$ (columns)

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Generating H_{11}

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for some $i \in I$. Hence $H_{11} = \{e_{11} e_{i\lambda} e_{11} : i \in I, \lambda \in \Lambda\}$ and

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Biordered sets

Let $E = E(S)$.

Definition For $e, f \in E$

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

We have that $\leq_{\mathcal{R}}$ $(\leq_{\mathcal{L}})$ is a preorder with associated equivalence $\mathcal{R}$ $(\mathcal{L})$.

Note If $e \leq_{\mathcal{R}} f$ then $fe = e$ and

$$(ef)(ef) = e(fe)f = eef = ef,$$

so that $ef \in E$. We say that

$$ef, fe \textbf{ are basic products } \text{ if } e \leq_{\mathcal{R}} f, f \leq_{\mathcal{R}} e, e \leq_{\mathcal{L}} f \text{ or } f \leq_{\mathcal{L}} e.$$

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Biordered sets

1. Under basic products, E satisfies a number of axioms; if S is regular, an extra axiom holds.
2. A **biordered set** is a partial algebra satisfying these axioms; if the extra one also holds it is a **regular biordered set**.
3. A biordered set is regular if and only if $E = E(S)$ for a regular semigroup S **Nambooripad (1979)**.
4. The category of inductive groupoids whose set of identities form a regular biordered set is equivalent to the category of regular semigroups **Nambooripad (1979)**.
5. Any biordered set E is $E(S)$ for some semigroup S **Easdown (1985)**.

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Free idempotent generated semigroups

Let E be a biordered set. We can assume $E = E(S)$ for a semigroup S. Let

$$\overline{E} = \{\overline{e} : e \in E\}$$

and let

$$\overline{E}^{+} = \{\overline{e_1} \dots \overline{e_n} : n \geq 1, e_i \in E\},$$

the **free semigroup** on $\overline{E}$.

Definition

$$\begin{aligned} IG(E) &= \langle \overline{E} : \overline{e}\,\overline{f} = \overline{ef}, ef \text{ is a basic product} \rangle \\ &= \overline{E}^{+} / \langle (\overline{e}\,\overline{f}, \overline{ef}) : ef \text{ is a basic product} \rangle. \end{aligned}$$

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Free idempotent generated semigroups

Facts

1. $\mathrm{IG}(E) = \langle \overline{E} \rangle$.
2. If $S = \langle E \rangle$, then $\overline{e} \mapsto e$ is an onto morphism.
3. $E(\mathrm{IG}(E)) = \overline{E} \cong E$, as a biordered set.
4. The above morphism is one-one on the set of $\mathcal{R}$-classes and $\mathcal{L}$-classes within a regular $\mathcal{D}$-class.

In view of the above, $\mathrm{IG}(E)$ is called the **free idempotent generated** semigroup on E.

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Maximal subgroups of free idempotent generated semigroups

These groups are homotopy groups of complexes determined by idempotent sequences and **singular squares**.

For $M_n(D)$ the $\leq_{\mathcal{R}}$ and $\leq_{\mathcal{L}}$ orders are given by **subspace inclusion**.

Putcha and Renner's theory of algebraic monoids shows that for connected reductive monoids the biorder is intimately related to the Tits building of the group of units.

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What groups can arise as maximal subgroups of IG(E)?

1. All such groups discovered in 1970s, 80s, 90s were free.
2. 2002 it was formally conjectured that all such groups were free.
3. 2009 **Brittenham, Margolis and Meakin** found $\mathbb{Z} \oplus \mathbb{Z}$ as a maximal subgroup of some IG(E).

Result of Gray and Ruskuc

Gray, Ruskuc (2012) show that **any** group can arise in this way.

1. Ruskuc **(1999)** found a presentation for H_e, where $e \in S$ and S is given by a presentation.
2. Gray and Ruskuc **(2012)** develop this to give presentations for $H_{\overline{e}}$ in $\mathrm{IG}(E)$.
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A current popular pastime

Question Given an idempotent generated semigroup S, find the groups $H_{\overline{e}}$ in $\mathrm{IG}(E)$. More particularly, find when $H_{\overline{e}} \cong H_e$.

1. 2010 **Brittenham, Margolis and Meakin** If $A \in E(M_n(D))$ and $\operatorname{rank} A = 1$, then $H_{\overline{A}} \cong H_A \cong D^*$.
2. 2012 **Gray, Ruskuc** If $e \in E(\mathcal{T}_n)$ and $\operatorname{rank} e = r \leq n-2$, then $H_{\overline{e}} \cong H_e \cong \mathcal{S}_r$.
3. 2014 **Dolinka, Gray** If $A \in E(M_n(D))$ and $\operatorname{rank} A = r < n/3$, then $H_{\overline{A}} \cong H_A \cong \mathrm{GL}(r, D)$.

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A current popular pastime - why not join in?

Observation Sets and vector spaces are examples of independence algebras.

Question Let A be an independence algebra with $\operatorname{rank} A = n$ and let

$$S = S(\operatorname{End}(A)) = \{\alpha \in \operatorname{End} A : \operatorname{rank} \alpha < n\} = \langle E \rangle.$$

If $e \in E(S)$ with $\operatorname{rank} e = n - 1$, then $H_{\overline{e}}$ is free.

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An easy way to get any group in an IG(E)
G and Dandan Yang

Theorem: G and Dandan Yang Let $A = Gx_1 \cup \ldots Gx_n$ be a free G-act of rank $n \geq 3$, and let $e \in \operatorname{End} A$ be a **rank 1** idempotent. Then

$$H_{\bar{e}} \cong H_e.$$

Recall $H_e \cong G$!

Corollary Any group occurs as a maximal subgroup of some IG(E).

Note Our proof is **very** easy; we do not need presentations; we use a natural biordered set.

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Going upwards

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Corollary Taking G to be trivial we obtain the Gray/Ruškuc result for $\mathcal{T}_n$.

Corollary Taking $n = 1$ we obtain any group (again!)

Where next?

Let A be an independence algebra of rank $n \geq 3$ and $e \in \text{End}(A)$ with rank $e \leq n-2$.

When do we have

$$H_{\overline{e}} \cong H_e?$$

1. True for rank $e = 1$, A has no constants **G and Yang**.
2. True for $A = V_4(D)$, rank $e = 2$ **Yang et al**
3. Conjectured not true for $A = V_6(\mathbb{Z}_2)$, rank $e = 2$ **O'Brien et al**

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